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HALPERN ITERATION PROCESS FOR APPROXIMATING SOLUTIONS OF MONOTONE YOSIDA VARIATIONAL INCLUSION, MINIMIZATION AND FIXED POINT PROBLEMS.

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Abstract. In this paper, we introduce an Halpern type iterative algorithm to approximate a common solution of monotone yosida variational inclusion problem, minimization problem and fixed point problem of finite family of quasi pseudo-contractive mappings in the framework of real Hilbert space. Using our iterative algorithm, we prove a strong convergence theorem of the aforementioned problems without imposing the compactness condition on our mapping. W also give an application of our main result to variational inequality problem. The result discuss in this paper extends and complements some related results in literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H, a point $p \in C$ is called a fixed point of a mapping $T : C \to C$, if $Tp = p$. We denote by $F(T)$, the set of all fixed points of T.

Definition 1.1. A nonlinear mapping $T : C \to C$ is said to be:
(i) Nonexpansive, if

$$||Tx - Ty|| \le ||x - y||, \ \forall\ x, y \in C;$$

(ii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$||Tx - p|| \le ||x - p||, \ \forall\ x \in C \text{ and } p \in F(T);$$

(iii) firmly quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$||Tx - p||^2 \le ||x - p||^2 - ||(I - T)x||^2, \ \forall\ x \in C \text{ and } p \in F(T);$$

(iv) strictly pseudo-contractive if there exists $\theta \in [0, 1)$ such that

$$||Tx - Ty||^2 \le ||x - y||^2 + \theta||(I - T)x - (I - T)y||^2, \ \forall\ x, y \in C;$$

(v) demicontractive, if $F(T) \neq \emptyset$ and there exists $\theta \in [0, 1)$ such that

$$||Tx - p||^2 \le ||x - p||^2 + \theta||Tx - x||^2, \ \forall\ x \in C \text{ and } p \in F(T);$$

(vi) monotone, if

$$\langle Tx - Ty, x - y\rangle \ge 0, \ \forall\ x, y \in C;$$

(vii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha||x - y||^2, \ \forall\ x, y \in C;$$

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(viii) α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||Tx - Ty||^2, \;x, y \in C.$$

In 2015, Chang et. al. introduced a nonlinear mapping called quasi-pseudocontractive mappings. This class of mappings includes the class of demicontractive, quasi-nonexpansive , strictly pseudo-contractive mappings with fixed points as a special cases. They gave the following definition for this class of mappings.

Definition 1.2. [4] A mapping $T : C \to C$ is said to be quasi pseudo-contractive mapping if $F(T) \neq \emptyset$ and

$$||Tx - p||^2 \leq ||x - p||^2 + ||Tx - x||^2, \;\forall\; x \in C \text{ and } p \in F(T).$$

Using this class of mappings, Chang et. al. [4] introduced an iterative algorithm to approximate the solutions of split equality fixed point problem of quasi pseudo-contractive mappings in real Hilbert space. They obtain a strong convergence result by imposing a compactness condition on their mapping.

Definition 1.3. Let D be a convex subset of a vector space E and $f : D \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,
(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in D$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);$$

(ii) f is called proper if there exists at least one $x \in D$ such that

$$f(x) \neq +\infty;$$

(iii) f is lower semicontinuous at $x_0 \in D$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

The theory of classical variational inequalities is one of the most important aspect in fixed point theory. Many other optimization problems such as Varaitional Inclusion Problem (VIP) and Monotone Variational Inclusion Problem (MVIP), to mention a few serves as a generalization to the classical variational inequality problem. The VIP which is to find an element $x \in H$ such that

$$0 \in Ax, \tag{1.1}$$

where $A : H \to 2^H$ is a multivalued mapping and H be a real Hilbert space covers many other optimization problems such as Minimization Problem (MP), Equilibrium Problem (EP), variational inequalities problem, to mention a few. A large number of problems arising in science, engineering, finance, operational research and optimizations e.t.c. can be formulated as a VIP, (see for instance [5, 6, 7] and the references contained in). Another generalization of the VIP is the MVIP which is to find an element $x \in H$ such that

$$0 \in A(x) + f(x), \tag{1.2}$$

where $A : H \to 2^H$ is a multivalued mapping and $f : H \to H$ is a single-valued mapping. It is obvious that MVIP (1.2) reduces to VIP (1.1) if $f \equiv 0$.
A mapping $A : H \to H$ is said to be maximal monotone if and only if its graph $gra(A) = \{(x, y) \in H \times H : y \in A(x)\}$ is not properly contained in the graph of any other monotone operator. It is well-known that when A is maximal monotone, then for each $x \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that $x \in (I + \lambda A)z$. Therefore, we denote by $J^A_\lambda := (I + \lambda A)^{-1}$ the resolvent operator of A of parameter λ.

In 2014, Wongvisarut and Kangtunyakarn [9] modified (1.2) as follows: find $x \in H$ such that

$$0 \in \sum_{i=1}^{N} a_i f_i u + Au, \tag{1.3}$$

where $f_i : H \to H$ is a single-valued mapping, $A : H \to 2^H$ be a multivalued mapping and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$. They denote the solution set of (1.3) by $VI(H, \sum_{i=1}^{N} a_i, f_i, A)$. They introduced an iterative scheme for finding a common element of the set of fixed points of a k-strictly pseudononspreading mapping, the set of solutions of a finite family of VIP and the set of solutions of a finite family of EP. They proved the following theorem:

Theorem 1.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and $F : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1)-(A4)$ (see [9] for conditions (A1)-(A4)). Let $M : H \to 2^H$ be a multivalued maximal monotone, then for every $i = 1, 2, ..., N$, let $A_i : H \to H$ be η_i-inverse strongly monotone with $\eta = \min_{i=1,2,...,N}\{\eta_i\}$ and $B_i : H \to H$ be μ_i-inverse strongly monotone with $\mu = \min_{i=1,2,...,N}\{\mu_i\}$. Let $\{T\}_{i=1}^{N}$ be a finite family of k_i-strictly pseudononspreading mappings of H into itself with $k = \max_{i=1,2,...N}\{k_i\}$. The sequence $\{x_n\}$ is generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \langle \sum_{i=1}^{N} b_i B_i x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in C; \\ y_n = \delta_n x - n + (1-\delta_n) J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i) u_n; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} c_i (I - \rho_n (I - T_i)) y_n, \ \forall\, n \in \mathbb{N}; \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $(0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n = 1, 0 \leq a_i, b_i, c_i \leq 1$, for every $i = 1, 2, ..., N, 0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1,\ r_n \in [g, h] \subseteq (0, 2\mu)$, and $\rho_n \in (0, 1-k)$ for all $n \geq 1$. Assume that the following conditions holds:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = 0$;
(ii) $0 < \lambda < 2\eta$;
(iii) $\sum_{n=1}^{\infty} \rho_n < \infty$;
(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = \sum_{i=1}^{N} c_i = 1$;
(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,\ \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$;
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty,\ \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,\ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. *Then, the sequence $\{x_n\}$ converges strongly to $x - 0 = P_\Gamma f(x_0)$, where $\Gamma := \bigcap_{i=1}^{N} F(T_i) \bigcap \bigcap_{i=1}^{N} VI(H, A_i, M)$ $\bigcap EP(F, \sum_{i=1}^{N} b_i B_i) \neq \emptyset$.*

In 2017, Khuangsatung and Kangtunyakarn [9] introduced an iterative algorithm to approximate a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of solutions of a finite family of VIP in real Hilbert space. They also proved the following theorem:

Theorem 1.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multivalued maximal monotone mapping and $A_i : H \to H$ be ν_i-inverse strongly monotone mapping with $\eta = \min_{i=1,2,...,N}\{\eta_i\}$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of H into itself with $\Gamma := \bigcap_{i=1}^{N} F(T_i) \bigcap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let $f : H \to H$ be a contractive mapping with $\phi \in (0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in H$ and*

$$\begin{cases} z_n^i = b_n x_n + (1-b_n) T_i x_n, \ n \geq 1; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} \delta_i A_i) x - n + \gamma_n \sum_{i=1}^{N} a_i z_n^i; \end{cases}$$

for all $0 < a_i, \delta_i \leq 1, \lambda > 0$ and $n \in \mathbb{N}$. Let $\{b_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ and $\alpha_n + \beta_n + \gamma = 1$. Assume the following conditions holds:
(i) $\lim_{n\to\infty} \alpha - n = 0, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$ *and* $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$*;*
(iii) $0 < \lambda < 2\eta$*, where* $\eta = \min_{i=1,2,\dots,N}\{\eta_i\}$*;*
(iv) $0 < c, \beta_n \leq d < 1 foralln \geq 1,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*, and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$*;*
(v) $\sum_{i=1}^{N} a_i = 1 = \sum_{i=1}^{N} \delta_i$*. Then the sequence* $\{x_n\}$ *converges strongly to* $z = P_\Gamma F(z)$*.*

One of the most important problems in optimization theory and non-linear analysis is the problem of approximating solutions of Minimization Problem (MP) which is to find $x \in H$ such that

$$f(x) = \min_{y\in H} f(y), \tag{1.4}$$

where $f : H \to (-\infty, \infty]$ is a proper and convex function. We denote by $argmin_{y\in H} f(y)$ the set of all minimizers of f on H. For $\lambda > 0$, the Moreau-Yosida resolvent of f on H which is known as the proximal operator of f of order λ is defined as follows:

$$Prox_\lambda f(x) = argminy \in H\{f(y) + \frac{1}{2\lambda}||y - x||^2\},\ \forall\ x \in H. \tag{1.5}$$

Martinet [30] introduced the well known proximal point algorithm which is a powerfool tool to approximate solution of (1.4). Since then, many authors have considered using different iterative scheme to approximate solutions of (1.4), See ([24, 12] and the references contained in).
Zarantonello [13] and Minty [14] introduced the notion of monotone operators, and since then many other researchers have shown significant interest because of the firm relation it has with the following evolution equation.

$$\begin{cases} \frac{dx}{dt} + A(x) = 0; \\ x(0) = x_0; \end{cases} \tag{1.6}$$

which is the model of many physical problems of practical applications. If the function A in (1.6) is not continuous, then it will be very difficult to solve these types of models. To solve these problem, Yosida introduced a natural step which is to find a sequence of lipschitz functions that approximate A in some sense. It is well known that two quite useful single-valued lipschitz continuous operators can be associated with a monotone operators, namely its resolvent operator and its Yosida approximation operator. The Yosida approximation operators are useful to approximate solutions of VIP using resolvent operators. Recently, many authors engaged the Yosida approximation operators to study some VIP using different techniques, See for exmaple ([15, 16, 19]).
Very recently, Ahmad et. al. [20] introduced the following Yosida inclusion problem to find $x \in X$ such that

$$0 \in T_{M,\lambda}^{\mathcal{H}(.,.)}(x) + M(x),\ \lambda > 0; \tag{1.7}$$

where X is a smooth Banach space, and $B : X \to 2^X$ is a multivalued mapping, $T_{B,\lambda}^{\mathcal{H}(.,.)}$ is the generalized Yosida approximation operator defined by

$$T_{B,\lambda}^{\mathcal{H}(.,.)}(x) = \frac{1}{\lambda}\big[I - R_{B,\lambda}^{\mathcal{H}(.,.)}\big](x), \forall\ x \in X; \tag{1.8}$$

where I is the identity mapping on X and $R_{B,\lambda}^{\mathcal{H}(.,.)}$ is the resolvent operator associated with the mappings $\mathcal{H}(.,.)$ and B. They proved the following fixed point formulation of Yosida inclusion problem (1.7):

$$x = R_{B,\lambda}^{\mathcal{H}(.,.)}\Big[\mathcal{H}(A,B)x - \lambda T_{B,\lambda}^{\mathcal{H}(.,.)}(x)\Big],\ \forall\ x \in X, \lambda > 0;$$

where $\mathcal{H}(A, B)$ is α-strongly accretive with respect to A, β-relaxed acrretive with respect to B and $\alpha > \beta$. See [20] for more details.

In 2018, Rahaman et. al. [21] introduced the Monotone Yosida Variational Inclusion Problem (MYVIP) which is to find $x \in H$ such that

$$0 \in f(x) + B(x) - T_\lambda^B(x); \tag{1.9}$$

where $T_\lambda^B = \frac{1}{\lambda}(I - R_\lambda^B)$ is the Yosida approximation operator of the mapping B, $R_\lambda^B = (I + \lambda B)^{-1}$ is the resolvent operator of the mapping B for $\lambda > 0$ and I is the identity mapping on a real Hilbert space H. We denote by Ω the solution set of (1.9). It has been shown in [21] that MYVIP (1.9) has a solution x if and only if $x = R_\lambda^B\big[I + \lambda(T_\lambda^B - f)\big]x$.

Remark 1.6. The MYVIP (1.9) generalizes the problems defined in (1.1), (1.2) and (1.3). See [21].

Remark 1.7. It is well known that R_μ^B is firmly nonexpansive, and hence it is averaged. Since the composition of averaged mappings is again average, therefore $R_\mu^B\big[I + \mu(T_\mu^B - g)\big]$ is average. Moreso, since every average mapping is strongly nonexpansive, hence we have that $R_\mu^B\big[I + \mu(T_\mu^B - g)\big]$ is strongly nonexpansive. See [21].

They proved a weak and strong convergence theorem of split type of MYVIP (1.9) using demicontractive property, nonexpansive property and strongly positive bounded linear property of mappings.

Motivated by the works of the aforementioned authors, we introduce an iterative algorithm to approximate the common solution of MYVIP (1.9), MP (1.4) and fixed point problem of a finite family of quasi pseudo-contractive mappings in the framework of real Hilbert space. We prove a strong convergence theorem of the aforementioned problems without imposing the compactness condition on either the mappings or the iterative scheme. Lastly, we give an application of our result to minimization problem. Our result extends the work of [8], [9], [4] and other related works in literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Definition 2.1. Let H be a real Hilbert space and C be a nonempty closed and convex of H. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to x with $||x_n - Tx_n|| = 0,\ Tx = x$.

Lemma 2.2. [27] *Let H be a Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0, 1)$, we have*

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

Lemma 2.3. [22] *Let C be a nonempty, closed and convex subset of real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to H$ be a finite family of quasi-nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $0 < a_i < 1$ with $\sum_{i=1}^N a_i = 1$. Then*

$$\bigcap_{i=1}^N F(T_i) = F\Big(P_C(I - \lambda\Big(\sum_{i=1}^N a_i(I - T_i)\Big)\Big).$$

Lemma 2.4. [17] *Let H be areal Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x - y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 + f(J_\lambda x) \le f(y).$$

Lemma 2.5. [11] *Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $0 < \lambda \leq \mu$ and $x \in \mathbb{N}$, we have*

$$||J_\lambda x - x|| \leq |J_\mu x - x||.$$

Lemma 2.6. [4] *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote $K := (1-\xi)I + \xi T\big((1-\eta)I + \eta T\big)$ if $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions holds.*
(1) $F(T) = F(T((1-\eta)I + \eta T)) = F(K)$;
(2) If T is demiclosed at 0, then K is also demiclosed at 0;
(3) In addition, if $T : H \to H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive, that is,

$$||Kx - u^*|| \leq ||x - u^*|| \ \forall \ x \in H \text{and } u^* \in F(T) = F(K).$$

Lemma 2.7. [22] *let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of quasi-nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then*

$$\Big|\Big|P_C\Big(I - \bar{\lambda}\Big(\sum_{i=1}^N k_i(I - T_i)\Big)\Big)x - z\Big|\Big|^2 \leq ||x - z||^2,$$

for all $x \in C$, where $0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$ and $0 < \bar{\lambda} < 1$.

Lemma 2.8. [26] *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n, \ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^\infty \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^\infty |\sigma_n\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $f : H \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $T_i : H \to H, i = 1, 2, ...N$ be finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed at the origin. Let $g : H \to H$ be inverse strongly monotone mapping and $B : H \to 2^H$ be a multivalued maximal monotone mapping with nonempty value. Assume that $\Gamma := \cap_{i=1}^N F(T_i) \cap argmin_{y\in H} f(y) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated iteratively for an arbitrary $u, x_1 \in H$ by*

$$\begin{cases} w_n = Prox_{\lambda_n f}x_n; \\ y_n = \alpha_n x_n + (1-\alpha_n)R_\mu^B\big[I + \mu(T_\mu^B - g)\big]w_n; \\ x_{n+1} = \beta_n u + \delta_n x_n + t_n(P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n, \forall \ n \in \mathbb{N}; \end{cases} \tag{3.1}$$

converges strongly to a point $x^* = P_\Gamma u$, where P_Γ is the metric projection of H onto Γ and $K_i := (1-\xi_n)I + \xi_n T_i\big((1-\eta_n)I + \eta_n T_i\big)$. Assume $0 < a_i < 1$ with $\sum_{i=1}^N a_i = 1$, $0 < \lambda \leq \lambda_n$ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{t_n\}$ are real sequences in $(0,1)$ such that $\beta_n + \delta_n + t_n = 1$ for all $n \in \mathbb{N}$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$;
(ii) $0 < b < \xi_n < \eta_n < c < \frac{1}{1+\sqrt{1+L^2}}$ $\forall \ n \geq 1$;
(iii) $\sum_{n=1}^\infty \rho_n < \infty$ and $0 < \rho_n < 1$;
(iv) $0 < \liminf t_n \leq \limsup t_n < 1$.

Proof. Let $\overline{x} \in \Gamma$, $V = R^B_\mu\big[I+\mu(T^B_\mu - g)\big]$; then from (3.1), Lemma 2.2 and the nonexpansive property of V, we have that

$$
\begin{aligned}
||y_n - \overline{x}||^2 &= ||\alpha_n x_n + (1-\alpha_n)Vw_n - \overline{x}||^2 \\
&\leq \alpha_n||x_n - \overline{x}||^2 + (1-\alpha_n)||Vw_n - \overline{x}||^2 - \alpha_n(1-\alpha_n)||x_n - Vw_n||^2 \\
&\leq \alpha_n||x_n - \overline{x}||^2 + (1-\alpha_n)||w_n - \overline{x}||^2 \\
&\leq \alpha_n||x_n - \overline{x}||^2 + (1-\alpha_n)||Prox_{\lambda_n f}x_n - \overline{x}||^2 \\
&\leq \alpha_n||x_n - \overline{x}||^2 + (1-\alpha_n)||x_n - \overline{x}||^2 \\
&= ||x_n - \overline{x}||^2. \qquad (3.2)
\end{aligned}
$$

Using (3.1), (3.2) and Lemma 2.7, we have that

$$
\begin{aligned}
||x_{n+1} - \overline{x}|| &= ||\beta_n(u - \overline{x}) + \delta_n(x_n - \overline{x}) + t_n||(P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n - \overline{x})|| \\
&\leq \beta_n||u - \overline{x}|| + \delta_n||x_n - \overline{x}|| + t_n||(P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n - \overline{x}|| \\
&\leq \beta_n||u - \overline{x}|| + \delta_n||x_n - \overline{x}|| + t_n||y_n - \overline{x}|| \\
&\leq \beta_n||u - \overline{x}|| + (1-\beta_n)||x_n - \overline{x}|| \\
&\leq \max\{||u - \overline{x}||,\ ||x_n - \overline{x}||\} \\
&\ \vdots \\
&\leq \max\{||u - \overline{x}||,\ ||x_1 - \overline{x}||\}.
\end{aligned}
$$

Therefore, $\{x_n\}$ is bounded and consequently, $\{w_n\}, \{y_n\}$ and $\{Vw_n\}$ are bounded. From (3.1), (3.2) and Lemma 2.7, we have that

$$
\begin{aligned}
||x_{n+1} - \overline{x}||^2 &\leq \beta_n||u - \overline{x}||^2 + t_n||(P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n - \overline{x}||^2 \\
&+ \delta_n||x_n - \overline{x}||^2 - \delta_n t_n||x_n - (P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n||^2 \\
&\leq \beta_n||u - \overline{x}||^2 + \delta_n||x_n - \overline{x}||^2 + t_n||y_n - \overline{x}||^2 \\
&- \delta_n t_n||x_n - (P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n||^2 \\
&\leq \beta_n||u - \overline{x}||^2 + \delta_n||x_n - \overline{x}||^2 + t_n||x_n - \overline{x}||^2 \\
&- \alpha_n t_n(1-\alpha_n)||x_n - Vw_n||^2 - \delta_n t_n||x_n - (P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n||^2 \\
&= \beta_n||u - \overline{x}||^2 + (1-\beta_n)||x_n - \overline{x}||^2 \\
&- \alpha_n t_n(1-\alpha_n)||x_n - Vw_n||^2 - \delta_n t_n||x_n - (P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n||^2. \qquad (3.3)
\end{aligned}
$$

CASE 1: Assume that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - \overline{x}||\}_{n=n_0}^\infty$ is monotone decreasing, then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty} ||x_n - \overline{x}|| = \lim_{n\to\infty} ||x_{n+1} - \overline{x}||.$$

From (3.3), we have that

$$\alpha_n t_n(1-\alpha_n)||x_n - Vw_n||^2 \leq \beta_n||u-\overline{x}||^2 + (1-\beta_n)||x_n-\overline{x}||^2 - ||x_{n+1}-\overline{x}||^2.$$

Using conditions (i) and (iv) of (3.1), we obtain that

$$\lim_{n\to\infty} ||x_n - Vw_n|| = 0. \tag{3.4}$$

Also, from (3.3) and conditions (i) and (iv) of (3.1), we obtain that

$$\lim_{n\to\infty} ||x_n - (P_C(I-\rho_n(\sum_{i=1}^{N} a_i(I-K_i)))y_n|| = 0. \tag{3.5}$$

From (3.1) and (3.4), we have that

$$||y_n - x_n|| \leq (1-\alpha_n)||Vw_n - x_n|| = 0 \to \infty. \tag{3.6}$$

Using (3.1), we have that

$$||x_{n+1}-x_n|| = \beta_n||u-x_n|| + t_n||(P_C(I-\rho_n(\sum_{i=1}^{N} a_i(I-K_i)))y_n - x_n||,$$

then, using condition (i) and (3.5) that

$$\lim_{n\to\infty} ||x_{n+1}-x_n|| = 0. \tag{3.7}$$

From Lemma 2.4, we obtain that

$$\frac{1}{2\lambda_n}||w_n - p||^2 + \frac{1}{2\lambda_n}||x_n - p||^2 + \frac{1}{2\lambda_n}||x_n - w_n||^2 \leq f(p) - f(w_n).$$

Since $f(p) \leq f(w_n)$ for all $n \geq 1$, we obtain that

$$||w_n - \overline{x}||^2 \leq ||x_n - \overline{x}||^2 - ||w_n - x_n||^2. \tag{3.8}$$

Now, from (3.1), we have that

$$\begin{aligned} ||y_n - \overline{x}||^2 &\leq \alpha_n||x_n-\overline{x}||^2 + (1-\alpha_n)||V_n w_n - \overline{x}||^2 - \alpha_n(1-\alpha_n)||x_n - V_n w_n||^2 \\ &\leq \alpha_n||x_n-\overline{x}||^2 + (1-\alpha_n)||w_n-\overline{x}||^2. \end{aligned} \tag{3.9}$$

On substituting (3.8) into (3.9), we obtain that

$$\begin{aligned} ||y_n - \overline{x}||^2 &\leq \alpha_n||x_n-\overline{x}||^2 + (1-\alpha_n)[||x_n-\overline{x}||^2 - ||w_n - x_n||^2] \\ &= ||x_n-\overline{x}||^2 - (1-\alpha_n)||w_n-x_n||^2. \end{aligned} \tag{3.10}$$

On substituting (3.10) into (3.3), we have that

$$\begin{aligned} ||x_{n+1}-\overline{x}||^2 &\leq \beta_n||u-\overline{x}||^2 + \delta_n||x_n-\overline{x}||^2 + t_n[||x_n-\overline{x}||^2 - (1-\alpha_n)||w_n-x_n||^2] \\ &= \beta_n||u-\overline{x}||^2 + (1-\beta_n)||x_n-\overline{x}||^2 - (1-\alpha_n)t_n||w_n-x_n||^2. \end{aligned} \tag{3.11}$$

This implies from (3.11) that

$$(1-\alpha_n)t_n||w_n-x_n||^2 \leq \beta_n||u-\overline{x}||^2 + (1-\beta_n)||x_n-\overline{x}||^2 - ||x_{n+1}-\overline{x}||^2.$$

From conditions (i) and (iv) of (3.1), we have that

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{3.12}$$

Since $\lambda_n > \lambda > 0$, we obtain from Lemma 2.5 that

$$||Prox_{\lambda f}x_n - x_n|| \leq ||Prox_{\lambda_n f}x_n - x_n||,$$

which implies from (3.12) that

$$||Prox_{\lambda f}x_n - x_n|| = 0. \tag{3.13}$$

From (3.6) and (3.12), we have that

$$||y_n - w_n|| \leq ||y_n - x_n|| + ||w_n - x_n|| \to 0, \text{ as } n \to \infty. \tag{3.14}$$

Also, from (3.4) and (3.12), we have that

$$\lim_{n\to\infty} ||w_n - V w_n|| = 0. \tag{3.15}$$

From (3.5) and (3.6), it follows that

$$\lim_{n\to\infty} (P_C(I - \rho_n(\sum_{i=1}^{N} a_i(I - K_i)))y_n - y_n|| = 0. \tag{3.16}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x^*. By (3.6) and (3.12), we have that there exists subsequences $\{y_{n_j}\}$ of $\{y_n\}$ and $\{w_{n_j}\}$ of $\{w_n\}$ which converges weakly to x^*. Hence, using Lemma 2.6 (2), Lemma 2.3 and (3.16), we obtain that $x^* \in \bigcap_{i=1}^{\infty} F(T_i) = F(K_i)$. Since $Prox_{\lambda f}$ is a nonexpansive single-valued mapping, it follows from the demi-closedness principle and (3.12) that $x^* \in F(Prox_{\lambda f})$. Lastly, using (3.15) and the demiclosedness principle, we have that $x^* \in \Omega$. Therefore, we conclude that $x^* \in \Gamma$.

Next, we show that $\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle \leq 0$. Since $\{x_n\}$ is bounded, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle = \limsup_{j\to\infty}\langle u - x^*, x_{n_j+1} - x^*\rangle.$$

Since $\{x_{n_j}\} \to \omega$ and $||x_{n+1} - x_n|| \to 0$, as $n \to \infty$. We have that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle &= \limsup_{j\to\infty}\langle u - x^*, x_{n_j+1} - x^*\rangle\\ &= \langle u - x^*, \omega - x^*\rangle \leq 0.\end{aligned} \tag{3.17}$$

Now, from (3.2), (3.3) and Lemma 2.8, we have that

$$\begin{aligned}||x_{n+1} - x^*||^2 &= ||\beta_n u + \delta_n x_n + t_n(P_C(I - \rho_n(\sum_{i=1}^{N} a_i(I - K_i)))y_n - x^*||^2\\ &\leq \delta_n||x_n - x^*||^2 + t_n||y_n - x^*||^2 + 2\beta_n(1-\beta_n)\langle u - x^*, x_{n+1} - x^*\rangle\\ &\leq \delta_n||x_n - x^*||^2 + t_n||w_n - x^*||^2 + 2\beta_n(1-\beta_n)\langle u - x^*, x_{n+1} - x^*\rangle\\ &\leq \delta_n||x_n - x^*||^2 + t_n||x_n - x^*||^2 + 2\beta_n(1-\beta_n)\langle u - x^*, x_{n+1} - x^*\rangle\\ &\leq (1-\beta_n)||x_n - x^*||^2 + 2\beta_n(1-\beta_n)\langle u - x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

applying condition (i) of (3.1), (3.17) and Lemma 2.6, we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where $\Delta_n = 2\beta_n(1-\beta_n)\langle u - x^*, x_{n+1} - x^*\rangle$.

CASE 2: Assume that $\{||x_n - x^*||\}$ is not monotone decreasing. Let $\Upsilon = ||x_n - x^*||^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{j \in \mathbb{N} : j \leq n, \Upsilon_j \leq \Upsilon_{j+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}, \ n \geq n_0.$$

From (3.3), we have that

$$\begin{aligned}&\alpha_{\tau(n)}t_{\tau(n)}(1-\alpha_{\tau(n)})||x_{\tau(n)} - V_{\tau(n)}w_{\tau(n)}||^2\\ &\leq \beta_{\tau(n)}||u - x^*||^2 + (1-\beta_{\tau(n)})||x_{\tau(n)} - x^*||^2 - ||x_{\tau(n)+1} - x^*||^2,\end{aligned}$$

using condition (i) and (iv) of (3.1), we obtain that

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - V_{\tau(n)}w_{\tau(n)}|| = 0. \tag{3.18}$$

Following the same steps as in Case 1, we have that $(\{x_{\tau(n)}\})$ converges weakly to x^*. Now for all $n \geq n_0$

$$0 \leq ||x_{\tau(n)+1} - x^*||^2 - ||x_{\tau(n)} - x^*||^2$$
$$\leq (1-\beta_{\tau(n)})||x_{\tau(n)} - x^*||^2 + \beta_{\tau(n)}[\beta_{\tau(n)}||u - x^*||^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle u - x^*, x_{\tau(n)+1} - x^*\rangle].$$

Therefore, we have that

$$||x_{\tau(n)} - x^*||^2 \leq \beta_{\tau(n)}||u - x^*||^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle u - x^*, x_{\tau(n)+1} - x^*\rangle.$$

Thus,

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - x^*||^2 = 0,$$

and

$$\lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)} = \lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)+1}.$$

Moreso, for $n \geq n_0$, it is observed that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1} if n = \tau(n)$ (that is $\tau(n) < n$) because $\Upsilon_j > \Upsilon_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently, for all $n \geq n_0$;

$$0 < \Upsilon_n \leq \max\{\Upsilon_{\tau(n)}, \Upsilon_{\tau(n)+1}\} = \Upsilon_{\tau(n)+1}.$$

Thus $\lim_{n\to\infty} \Upsilon_n = 0$. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T_i : H \to H, i = 1, 2, ...N$ be finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed at the origin. Let $g : H \to H$ be inverse strongly monotone mapping and $B : H \to 2^H$ be a multivalued maximal monotone mapping with nonempty value. Assume that $\Gamma := \cap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated iteratively for an arbitrary $x_1 \in H$ by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) R_\mu^B\big[I + \mu(T_\mu^B - g)\big]x_n; \\ x_{n+1} = \beta_n u + \delta_n x_n + t_n(P_C(I - \rho_n(\sum_{i=1}^N a_i(I - K_i)))y_n, \forall\ n \in \mathbb{N} \end{cases} \tag{3.19}$$

converges strongly to a point $\overline{x} = P_\Gamma u$, where P_Γ is the metric projection of H onto Γ and $K_i := (1-\xi_n)I + \xi_n T_i\big((1-\eta_n)I + \eta_n T_i\big)$. Assume $0 < a_i < 1$ with $\sum_{i=1}^N a_i = 1$ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{t_n\}$ are real sequences in $(0, 1)$ such that $\beta_n + \delta_n + t_n = 1$ for all $n \in \mathbb{N}$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^\infty \beta_n = \infty$;
(ii) $0 < b < \xi_n < \eta_n < c < \frac{1}{1+\sqrt{1+L^2}}\ \forall\ n \geq 1$;
(iii) $\sum_{n=1}^\infty \rho_n < \infty$ *and* $0 < \rho_n < 1$;
(iv) $0 < \liminf t_n \leq \limsup t_n < 1$.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $f : H \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $T : H \to H, i = 1, 2, ...N$ be a nonexpansive mapping. Let $g : H \to H$ be inverse strongly monotone mapping and $B : H \to 2^H$ be a multivalued maximal monotone mapping with nonempty value. Assume that $\Gamma := F(T) \cap argmin_{y\in H} f(y) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated iteratively for an arbitrary $x_1 \in H$ by*

$$\begin{cases} w_n = Prox_{\lambda_n f} x_n; \\ y_n = \alpha_n x_n + (1-\alpha_n) R_\mu^B\big[I + \mu(T_\mu^B - g)\big]w_n; \\ x_{n+1} = \beta_n u + \delta_n x_n + t_n T y_n, \forall\ n \in \mathbb{N} \end{cases} \tag{3.20}$$

converges strongly to a point $\overline{x} = P_\Gamma u$, where P_Γ is the metric projection of H onto Γ. Assume $0 < \lambda \leq \lambda_n$ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{t_n\}$ are real sequences in $(0, 1)$ such that $\beta_n + \delta_n + t_n = 1$ for all $n \in \mathbb{N}$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$;
(ii) $0 < \liminf t_n \leq \limsup t_n < 1$.

Conclusion We introduce an iterative algorithm to approximate common solutions of monotone yosida variational inclusion problem, minimization problem and fixed point problem of a finite family of quasi pseudo-contractive mappings in the framework of real Hilbert space. The monotone yosida variational inclusion problem considered in this paper generalizes monotone variational inclusion problem and variational inclusion problem. We prove a strong convergence result without imposing the compactness condition. Lastly, we give an application to variational inequality problem.

4. **Application to Variational Inequality Problem**

Let H be a real Hilbert space and h be a proper, convex and lower semi-continuous function of H into $\mathbb{R}$. Then the subdifferential δh of h is defined as follows:

$$\delta h(x) = \{z \in H : h(x) + \langle z, u - x\rangle \leq h(u), \forall\ u \in H\}, \tag{4.1}$$

for all $x \in H$. We know that δh is a maximal monotone operator (see [18]). Let C be a nonempty, closed and convex subset of H and ι_C be the indicator function of C which is defined by

$$\iota_C = \begin{cases} 0,\ if\ x \in C, \\ +\infty,\ if\ x \notin C. \end{cases}$$

Then, ι_C is a proper, convex and lower semicontinuous function on H. So, we can define the resolvent operator as $R_\lambda^{\delta_{\iota_C}}$ of δ_{ι_C} for $\lambda > 0$, i.e.

$$R_\lambda^{\delta_{\iota_C}}(x) = (I + \lambda\delta_{\iota_C})^{-1}(x),\ x \in H.$$

We know that $R_\lambda^{\delta_{\iota_C}}(x) = P_C(x)$ for all $x \in H$ and $\lambda > 0$ (see [**?**]). Moreover, for a single-valued operator $f : H \to H$, we have that

$$x \in (f + \delta_{\iota_C})^{-1}(0) \Longleftrightarrow x \in VI(C, f).$$

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $x \in H$, it is known that there exists a unique element $P_C x$ of C such that

$$||x - P_C x|| = \inf\{||x - y|| : y \in C\},$$

where P_C is referred to as the nearest point mapping or metric projection from H onto C. Using the application stated below, iterative algorithm (3.1) becomes

$$\begin{cases} w_n = Prox_{\lambda_n f} x_n; \\ y_n = \alpha_n x_n + (1 - \alpha_n)P_C(I - \lambda_n f)u_n; \\ x_{n+1} = \beta_n u + \delta_n x_n + t_n(P_C(I - \rho_n(\sum_{i=1}^{N} a_i(I - K_i)))y_n, \forall\ n \in \mathbb{N}. \end{cases}$$

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